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Solitons in Brane Worlds II

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Abstract

We study the solution describing a non-extreme dilatonic $(p+1)$-brane intersecting a D-dimensional extreme dilatonic domain wall, where one of its longitudinal directions is along the direction transverse to the domain wall, in relation to the Randall-Sundrum type model. The dynamics of the probe $(p+1)$-brane in such source background reproduces that of the probe p-brane in the background of the $(D-1)$-dimensional source p-brane. However, as for a probe test particle, the dynamics in one lower dimensions is reproduced, only when the source $(p+1)$-brane is uncharged.

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1 Introduction

Since it was shown [1] by Randall and Sundrum (RS) that gravity in the background of non-dilatonic domain wall with the exponentially decreasing warp factor is effectively compactified, some efforts have been made to understand gravitating objects living in such domain wall, e.g., Refs. [2, 3, 4, 5, 6]. We showed [7] that dilatonic domain walls also effectively compactify gravity if the dilaton coupling parameter is sufficiently small. The main motivation to consider dilatonic domain walls was that the consistency of equations of motion requires the cosmological constant term to have dilaton factor in order for the domain wall spacetime to admit *charged* brane solutions.

In our previous work [7], we constructed completely localized solutions describing extreme charged branes living in the worldvolume of extreme dilatonic domain walls for the purpose of understanding charged branes in the RS type model. Unexpectedly, it is found out [8] that a charged p-brane is not effectively compactified to the charged p-brane in one lower dimensions. It is speculated [8] that this is due to the unusual properties of the Kaluza-Klein (KK) modes of gauge fields and also presumably form fields that the zero mode is not localized on the lower-dimensional hypersurface of the domain wall [9] and the massive modes strongly couple to the fields on the brane [10, 9]. So, following the result of Ref. [3] that the Schwarzschild black hole in four-dimensional world within a domain wall should be regarded as an uncharged black string in five dimensions, we speculated [8] that a charged p-brane in one lower dimensions might have to be regarded as a charged $(p+1)$-brane where one of its longitudinal directions is along the transverse direction of the domain wall.

It is the purpose of this paper to construct a solution describing a non-extreme charged dilatonic $(p+1)$-brane in an extreme dilatonic domain wall in D dimensions where one of the longitudinal directions of the brane is along the transverse direction of the domain wall and to study its properties in relation to the RS type model. We find that in the case of an uncharged branes, physics of the uncharged p-brane in one lower dimensions is reproduced by the uncharged $(p+1)$-brane in the domain wall. However, when the $(p+1)$-brane is charged, the dynamics of a test particle in the background of the charged p-brane in one lower dimensions is not reproduced. This is due to the fact that generally the transverse (to the domain walls) component of the spacetime metrics of charged branes in the domain walls has non-trivial dependence on the longitudinal coordinates of the domain walls. Note, the original RS model [11, 1, 12] assumes that the perturbations of the domain wall metric should be along the longitudinal directions of the domain wall, only, in order for the lower-dimensional gravity to be reproduced. So, in order for the RS type model to admit wide variety of gravitating objects which reproduce physics in one lower dimensions, one has to somehow modify the model.

The paper is organized as follows. In section 2, we present the solution describing a nonextreme dilatonic $(p+1)$-brane intersecting an extreme dilatonic domain wall in D-dimensions where one of the longitudinal directions of the brane is along the direction transverse to the domain wall. In section 2, we study the dynamics of the probe $(p+1)$-brane in such source background, comparing with the dynamics of the probe p-brane in the background of the $(D-1)$-dimensional source p-brane. In section 3, we repeat the same analysis with a test particle.

2 Brane-World Solitons

In this section, we discuss the D-dimensional solution describing a non-extreme dilatonic $(p+1)$-brane with the dilaton coupling parameter a_{p+1} intersecting extreme dilatonic domain wall with the dilaton coupling parameter a such that one of the longitudinal directions of the $(p+1)$-brane is along the direction transverse to the domain wall. The configuration is given in the following table.

	t	$\mathbf{w}$	$\mathbf{x}$	y
brane	$\bullet$	$\bullet$		$\bullet$
domain wall	$\bullet$	$\bullet$	$\bullet$	

Here, t is the time coordinate, $\mathbf{w}=(w_1,...,w_p)$ and y are the longitudinal coordinates of the $(p+1)$-brane, and $\mathbf{w}$ and $\mathbf{x}=(x_1,...,x_{D-p-2})$ are the longitudinal coordinates of the domain wall. The solution for such configuration solves the equations of motion of the following action:

$$S=\frac{1}{2\kappa_D}\int dx^D\sqrt{-g}\left[\mathcal{R}-\frac{4}{D-2}(\partial\phi)^2-\frac{1}{2\cdot(p+3)!}e^{2a_{p+1}\phi}F^2_{p+3}+e^{-2a\phi}\Lambda\right]. \quad (1)$$

The solution has the following form:

$$\begin{aligned}
ds^2 &= H^{\frac{4}{(D-2)\Delta}}\left[H_{p+1}^{-\frac{4(D-p-4)}{(D-2)\Delta_{p+1}}}\left(-fdt^2+dw_1^2+\cdots+dw_p^2\right)\right.\\
&\quad\left.+H_{p+1}^{\frac{4(p+2)}{(D-2)\Delta_{p+1}}}\left(f^{-1}dx^2+x^2d\Omega^2_{D-p-3}\right)\right]+H^{\frac{4(D-1)}{(D-2)\Delta}}H_{p+1}^{-\frac{4(D-p-4)}{(D-2)\Delta_{p+1}}}dy^2,\\
e^{2\phi} &= H^{\frac{(D-2)a}{\Delta}}H_{p+1}^{\frac{(D-2)a_{p+1}}{\Delta_{p+1}}},\\
A_{tw_1...w_py} &= \frac{2}{\sqrt{\Delta_{p+1}}}\frac{\mu\cosh\delta_{p+1}\sinh\delta_{p+1}}{x^{D-p-4}}H^{-1}_{p+1}H^{\frac{4}{\Delta}},
\end{aligned} \quad (2)$$

where the harmonic functions H_{p+1} and H for the $(p+1)$-brane and the domain wall, and the non-extremality function f are given by

$$H_{p+1}=1+\frac{\mu\sinh^2\delta_{p+1}}{x^{D-p-4}},\qquad H=1+Q|y|,\qquad f=1-\frac{\mu}{x^{D-p-4}}, \quad (3)$$

and the parameters Δ's in the solutions are defined as

$$\begin{aligned}\Delta_{p+1} &= \frac{(D-2)a_{p+1}^2}{2} + \frac{2(p+2)(D-p-4)}{D-2}, \\ \Delta &= \frac{(D-2)a^2}{2} - \frac{2(D-1)}{D-2}.\end{aligned} \qquad (4)$$

Here, $x \equiv |\mathbf{x}|$ is the radial coordinate of the transverse space of the $(p+1)$-brane, $\mu > 0$ is the non-extremality parameter, and Q is related to the cosmological constant Λ as $\Lambda = -2Q^2/\Delta$. The extreme limit of the $(p+1)$-brane is achieved by taking $\mu \to 0$ such that $\mu e^{2\delta_{p+1}}$ is a non-zero constant. The consistency of the equations of motion requires that the dilaton coupling parameters satisfy the following constraint:

$$a a_{p+1} = -\frac{4(D-p-4)}{(D-2)^2}. \qquad (5)$$

Note, this constraint (and the intersection rules arising from this constraint) is different from the ordinary one [13, 14, 15, 16] which is satisfied by the intersecting branes whose harmonic functions depend on the overall transverse coordinates. The configuration under consideration in this paper does not have an overall transverse direction and each constituent is localized along the relative transverse directions. So, the solution (2) can be regarded as a particular case of general semi-localized solutions describing intersecting branes in which each constituent is localized along the relative transverse directions and delocalized along the overall transverse directions. Such semi-localized intersecting brane solutions are constructed in Ref. [17] for the extreme case and in Ref. [18] for the non-extreme case, and satisfy different intersection rules [2]. An example is intersecting two NS5-branes with one-dimensional intersection [19, 20], rather than the three-dimensional one. It is interesting to note that when $p = D-4$ there is no constraint on the dilaton coupling parameters. However, this particular case is not interesting in our work because the harmonic function H_{p+1} is logarithmic for $p = D-4$.

3 Probing Brane-World Solitons

In this section, we repeat the probe dynamics analysis of our previous work [8] with the source background (2) of a brane-world soliton, comparing with the probe dynamics in the following source background of a $(D-1)$-dimensional non-extreme dilatonic p-brane:

$$ds^2 = H_p^{-\frac{4(D-p-4)}{(D-3)\Delta_p}} \left[-f dt^2 + dw_1^2 + \cdots + dw_p^2\right] + H_p^{\frac{4(p+1)}{(D-3)\Delta_p}} \left[f^{-1} dx^2 + x^2 d\Omega_{D-p-3}^2\right],$$

[2] No-force requirement [15] on the probe brane in the source brane background yields the ordinary intersection rules, only. So, the constraint (5) on the dilaton coupling parameters is different from the one we expected through the no-force requirement in our previous work [8].

$$e^{2\phi} = H_p^{\frac{(D-3)a_p}{\Delta_p}}, \qquad A_{tx_1\dots x_p} = \frac{2}{\sqrt{\Delta_p}} \frac{\mu\cosh\delta_p \sinh\delta_p}{x^{D-p-4}} H_p^{-1}, \tag{6}$$

where

$$\begin{aligned} H_p &= 1+\frac{\mu\sinh^2\delta_p}{x^{D-p-4}}, \qquad f = 1-\frac{\mu}{x^{D-p-4}}, \\ \Delta_p &= \frac{(D-3)a_p^2}{2} + \frac{2(p+1)(D-p-4)}{D-3}. \end{aligned} \tag{7}$$

We regard this solution as being obtained by compactifying a D-dimensional $(p+1)$-brane with a dilaton coupling parameter a_p (i.e., the solution (2) with $H=1$) along a longitudinal direction (which is the y-direction in the notation of Eq. (2)) on S^1. Since the parameter Δ_p is invariant under reductions or oxidations which do not involve field truncation, one can see that a_p is related to a_{p+1} of the solution in one higher dimension, namely that of the $(p+1)$-brane in Eq. (2), as

$$(D-3)a_p^2 = (D-2)a_{p+1}^2 + 4\frac{(D-p-4)^2}{(D-2)(D-3)}. \tag{8}$$

3.1 Probing with a $(p+1)$-brane

The worldvolume action for a dilatonic $(p+1)$-brane with the following bulk action:

$$S_E = \frac{1}{2\kappa_D^2}\int d^D x\sqrt{-g}\left[\mathcal{R} - \frac{4}{D-2}(\partial\phi)^2 - \frac{1}{2\cdot(p+3)!}e^{2a_{p+1}\phi}F_{p+3}^2\right] \tag{9}$$

has the following form:

$$\begin{aligned} S_\sigma &= -T_{p+1}\int d^{p+2}\xi \left[e^{-a_{p+1}\phi}\sqrt{-\det\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}}\right. \\ &\left. \quad + \frac{\sqrt{\Delta_{p+1}}}{2}\frac{1}{(p+2)!}\epsilon^{a_1\dots a_{p+2}}\partial_{a_1}X^{\mu_1}\dots\partial_{a_{p+2}}X^{\mu_{p+2}}A_{\mu_1\dots\mu_{p+2}}\right], \end{aligned} \tag{10}$$

where T_{p+1} is the tension of the probe $(p+1)$-brane and the target space fields $g_{\mu\nu}$, ϕ and $A_{\mu_1\dots\mu_{p+2}}$ are the background fields (produced by the source brane) in which the probe $(p+1)$-brane with the target space coordinates $X^\mu(\xi^a)$ $(\mu = 0,1,...,D-1)$ and the worldvolume coordinates ξ^a $(a=0,1,...,p+1)$ moves.

In the static gauge, in which $X^a = \xi^a$, the pull-back fields for the probe $(p+1)$-brane, oriented in the same way as the source $(p+1)$-brane, take the following forms:

$$\begin{aligned} \hat{G}_{ab} &\equiv g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu = g_{ab} + g_{ij}\partial_a X^i \partial_b X^j, \\ \hat{A}_{a_1\dots a_{p+2}} &\equiv A_{\mu_1\dots\mu_{p+2}}\partial_{a_1}X^{\mu_1}\dots\partial_{a_{p+2}}X^{\mu_{p+2}} = A_{a_1\dots a_{p+2}}, \end{aligned} \tag{11}$$

where the indices $i, j = 1, ..., D-p-2$ label the transverse space of the probe $(p+1)$-brane, i.e., $(X^i) = (x_1, ..., x_{D-p-2})$ in the notation of Eq. (2). So, the worldvolume action (10) takes the following form:

$$S_\sigma = -T_{p+1}\int d^{p+2}\xi \left[e^{-a_{p+1}\phi}\sqrt{-\det\left(g_{ab}+g_{ij}\partial_a X^i \partial_b X^j\right)} + \frac{\sqrt{\Delta_{p+1}}}{2}A_{01...p+1}\right]. \quad (12)$$

From now on, we assume that the target space transverse coordinates X^i for the probe $(p+1)$-brane depend on the time coordinate $\tau = \xi^0$ only, i.e., $X^i = X^i(\tau)$.

By substituting the explicit expressions (2) for the source background fields of the brane-world $(p+1)$-brane into the general expression (12) for the probe $(p+1)$-brane action, one obtains the following:

$$\begin{aligned} S_\sigma &= -T_{p+1}\int d^{p+2}\xi\, H^{\frac{4}{\Delta}}\left[H^{-1}_{p+1}f^{\frac{1}{2}}\sqrt{1-H_{p+1}^{\frac{4}{\Delta_{p+1}}}\left\{f^{-2}\left(\frac{dx}{d\tau}\right)^2 + x^2 f^{-1}\mu_m^2\left(\frac{d\phi_m}{d\tau}\right)^2\right\}}\right. \\ &\qquad \left. + \frac{\mu\cosh\delta_{p+1}\sinh\delta_{p+1}}{x^{D-p-4}}H^{-1}_{p+1}\right]. \end{aligned} \quad (13)$$

On the other hand, the probe p-brane action in the source background (6) of the $(D-1)$-dimensional p-brane is

$$\begin{aligned} S_\sigma &= -T_p\int d^{p+1}\xi \left[H^{-1}_{p}f^{\frac{1}{2}}\sqrt{1-H_{p}^{\frac{4}{\Delta_{p}}}\left\{f^{-2}\left(\frac{dx}{d\tau}\right)^2 + x^2 f^{-1}\mu_m^2\left(\frac{d\phi_m}{d\tau}\right)^2\right\}}\right. \\ &\qquad \left. + \frac{\mu\cosh\delta_{p}\sinh\delta_{p}}{x^{D-p-4}}H^{-1}_{p}\right]. \end{aligned} \quad (14)$$

Here, the angular coordinates $0 \leq \phi_m < 2\pi$ ($m = 1, ..., [(D-p-2)/2]$) are associated with $[(D-p-2)/2]$ rotation planes in the transverse space of the branes (with the coordinates $\mathbf{x}$) and the index m is summed over $m = 1, ..., [(D-p-2)/2]$. The remaining angular coordinates, which determine the direction cosines μ_m, are constant due to the conservation of the direction of the angular momentum.

We see that the probe actions (13) and (14) have the same form except that the probe $(p+1)$-brane action (13) has an additional overall factor $H^{4/\Delta}$. So, the dynamics of the probe $(p+1)$-brane in the background of the brane-world $(p+1)$-brane is identical to that of the probe p-brane in the background of the p-brane in one lower dimensions. (Note, $\Delta_p = \Delta_{p+1}$, provided the $(D-1)$-dimensional p-brane is obtained by compactifying the D-dimensional $(p+1)$-brane on S^1 without field truncation.) The effect of the overall factor $H^{4/\Delta}$ in the former case is to effectively increase [decrease] the tension of the probe $(p+1)$-brane when $\Delta > 0$ [$\Delta < 0$], namely $T^{\rm eff}_{p+1} = T_{p+1}H^{4/\Delta}$.

Since the probe actions for the two cases have the same form, one also expects that the first law of black brane thermodynamics of the p-brane in one lower dimensions can be extracted from that of the $(p+1)$-brane in the domain wall, and vice versa. The first law of black brane thermodynamics of the latter brane with the solution given by Eq. (2) is

$$\delta M_{p+1} = T_H^{p+1} \delta S_{p+1} + \Phi_{p+1} \delta Q_{p+1}, \tag{15}$$

where M_{p+1} and S_{p+1} are the ADM mass and the entropy of the source $(p+1)$-brane per unit $(p+1)$-brane worldvolume, Q_{p+1} is the source $(p+1)$-brane charge normalized to take integer values (i.e., the number of elementary $(p+1)$-branes with unit charge), and the Hawking temperature T_H^{p+1} and the chemical potential Φ_{p+1} of the source $(p+1)$-brane are given by

$$T_H^{p+1} = \frac{D-p-4}{4\pi \mu^{\frac{1}{D-p-4}} \cosh^{\frac{4}{\Delta_{p+1}}} \delta_{p+1}}, \qquad \Phi_{p+1} = \frac{2}{\sqrt{\Delta_{p+1}}} T_{p+1} H^{\frac{4}{\Delta}} \tanh \delta_{p+1}. \tag{16}$$

In the case of the source p-brane in $(D-1)$-dimensions with the solution given by Eq. (6), the temperature T_H^p and the chemical potential Φ_p are respectively related to those of the $(p+1)$-brane as $T_H^p = T_H^{p+1}$ and $\Phi_p/T_p = \Phi_{p+1}/(T_{p+1}H^{4/\Delta})$, if we let $\delta_p = \delta_{p+1}$. Note, Δ_{p+1} in Eq. (2) and Δ_p in Eq. (6) are the same, if the p-brane is obtained from the $(p+1)$-brane through the dimensional reduction without field truncation. One can think of the changes δM_{p+1}, δS_{p+1} and δQ_{p+1} as being due to an addition of the probe $(p+1)$-brane to the source $(p+1)$-brane [21]. Namely, we bring the probe $(p+1)$-brane with a unit brane charge from spatial infinity ($x = \infty$) to the source brane horizon ($x = x_H = \mu^{1/(D-p-4)}$). Then, one can interpret $T_H^{p+1}\delta S_{p+1}$ as the heat released by the probe $(p+1)$-brane while it falls inside the source $(p+1)$-brane, which is just the difference in static potential energy $V_{p+1}(x)$ of the probe $(p+1)$-brane, i.e., $T_H^{p+1}\delta S_{p+1} = V(\infty) - V(x_H)$ [21]. From the above probe actions (13) and (14), one obtains the following static potentials on the probe branes:

$$V_{p+1} = T_{p+1} H^{\frac{4}{\Delta}} \left(f^{\frac{1}{2}} + \frac{\mu \cosh \delta_{p+1} \sinh \delta_{p+1}}{x^{D-p-4}} \right) H_{p+1}^{-1}, \tag{17}$$

for the probe $(p+1)$-brane, and

$$V_p = T_p \left(f^{\frac{1}{2}} + \frac{\mu \cosh \delta_p \sinh \delta_p}{x^{D-p-4}} \right) H_p^{-1}, \tag{18}$$

for the probe p-brane. As expected, in the extreme limit ($\mu \to 0$ with $\mu e^{2\delta}$ finite constant), the potentials are constant in accordance with the no-force condition for extreme branes. We see that the two static potentials are related as $V_{p+1}/(T_{p+1}H^{4/\Delta}) = V_p/T_p$ and therefore $T_H^{p+1}\delta S_{p+1}/(T_{p+1}H^{4/\Delta}) = T_H^p \delta S_p/T_p$, if $\delta_p = \delta_{p+1}$. The probe actions (13) and (14) with $\delta_p = \delta_{p+1}$ imply that the mass density changes are related as $\delta M_{p+1}/(T_{p+1}H^{4/\Delta}) = \delta M_p/T_p$. Since the probe branes have unit charges, $\delta Q_p = 1 = \delta Q_{p+1}$. Gathering all the above, one can bring the first law of the black $(p+1)$-brane thermodynamics (15) to the following first law of thermodynamics of the black p-brane in $D-1$ dimensions:

$$\delta M_p = T_H^p \delta S_p + \Phi_p \delta Q_p. \tag{19}$$

3.2 Probing with a test particle

In analyzing the dynamics of a test particle in a curved spacetime background, it is convenient to utilize the symmetry of the spacetime. The Killing vectors of the spacetime metrics of both of the solutions (2) and (6) are $\partial/\partial t$, $\partial/\partial w_i$ and $\partial/\partial\phi_m$. Contracting these Killing vectors with the velocity $U^\mu = dx^\mu/d\lambda$ of the test particle along the geodesic path $x^\mu(\lambda)$ parametrized by an affine parameter λ, one obtains the following constants of motion for the test particle:

$$\begin{aligned}
E &= -g_{\mu\nu}\left(\frac{\partial}{\partial t}\right)^\mu U^\nu = -g_{tt}\frac{dt}{d\lambda}, \\
p^i &= g_{\mu\nu}\left(\frac{\partial}{\partial w_i}\right)^\mu U^\nu = g_{ii}\frac{dw_i}{d\lambda}, \\
J^m &= g_{\mu\nu}\left(\frac{\partial}{\partial \phi_m}\right)^\mu U^\nu = g_{\phi_m\phi_m}\frac{d\phi_m}{d\lambda}.
\end{aligned} \tag{20}$$

In addition, there is another constant of motion associated with metric compatibility along the geodesic path:

$$\epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}, \tag{21}$$

where $\epsilon = 1, 0$ respectively for a massive particle (i.e., a timelike geodesic) and a massless particle (i.e., a null geodesic).

For the simplicity of the calculation, we shift the transverse coordinate y of the domain wall so that the harmonic function for the domain wall takes the form $H = Qy$, where we restrict to the region $y \geq 0$. Then, we apply the following change of coordinate:

$$y = \left(\frac{\Delta+2}{\Delta}Q^{-\frac{2}{\Delta}}z\right)^{\frac{\Delta}{\Delta+2}} \tag{22}$$

to bring the domain wall metric to the conformally flat form. In this new coordinate, the metric in Eq. (2) takes the following form:

$$ds^2 = \left(\frac{\Delta+2}{\Delta}Qz\right)^{\frac{4}{(D-2)(\Delta+2)}}\left[H_{p+1}^{-\frac{4(D-p-4)}{(D-2)\Delta_{p+1}}}\left(-f dt^2 + dw_1^2 + \cdots + dw_p^2\right)\right.$$

$$+H_{p+1}^{\frac{4(p+2)}{(D-2)\Delta_{p+1}}}\left(f^{-1}dx^2+x^2d\Omega^2_{D-p-3}\right)+H_{p+1}^{-\frac{4(D-p-4)}{(D-2)\Delta_{p+1}}}dz^2\Bigg]. \tag{23}$$

For the test particle moving along the z-direction, i.e., only the z-component of U^μ is non-zero, the geodesic motion is described by the following equation, derived from the geodesic equation and Eq. (21):

$$\frac{d}{d\lambda}\left[z^{\frac{4}{(D-2)(\Delta+2)}}\frac{dz}{d\lambda}\right]=-\frac{2\epsilon}{(D-2)(\Delta+2)}\left(\frac{\Delta+2}{\Delta}Q\right)^{-\frac{4}{(D-2)(\Delta+2)}}H_{p+1}^{\frac{4(D-p-4)}{(D-2)\Delta_{p+1}}}\frac{1}{z}. \tag{24}$$

The geodesic path $z(\lambda)$ for a massless test particle (i.e., the $\epsilon=0$ case) is $z=$ constant or

$$z=z_0\lambda^{\frac{(D-2)(\Delta+2)}{(D-2)(\Delta+2)+4}}, \tag{25}$$

where z_0 is an arbitrary constant. The general explicit expression for the timelike geodesic path (i.e., the $\epsilon=1$ case) is hard to obtain. So, in the following we shall consider only the case of the null geodesic motion along the z-direction. The null geodesic path $z=$ constant simply corresponds to the motion constrained along the longitudinal directions of the domain wall.

Making use of the constants of the motion in Eq. (20), one can put Eq. (21) into the following form:

$$\left(\frac{dx}{d\lambda}\right)^2+\frac{g_{zz}}{g_{xx}}\left(\frac{dz}{d\lambda}\right)^2+\frac{J_m^2}{g_{xx}g_{\phi_m\phi_m}}+\frac{E^2}{g_{xx}g_{tt}}+\frac{\epsilon}{g_{xx}}=0, \tag{26}$$

where the index m is summed over $m=1,...,[(D-p-2)/2]$. The angular coordinates of the transverse space of the branes (with the coordinates $\mathbf{x}$) in the spherical coordinates, except for the ones ϕ_m associated with the angular momenta J_m, are constant due to the conservation of the direction of the angular momentum. And we are considering the motion of the test particle with the longitudinal coordinates w_i of the branes constant, which is possible due to the conservation of the linear momenta p^i along those directions. By plugging the explicit expression (23) for the background metric of the source brane into the general expression (26), we obtain

$$\begin{aligned}\left(\frac{dx}{d\lambda}\right)^2 &+ \mathcal{C}^{-2}\left[\left\{\mathcal{C}^2H_{p+1}^{-\frac{4}{\Delta_{p+1}}}\left(\frac{dz}{d\lambda}\right)^2+\frac{\mathcal{J}^2}{H_{p+1}^{\frac{8(p+2)}{(D-2)\Delta_{p+1}}}x^2}\right\}f-E^2H_{p+1}^{\frac{4(D-2p-6)}{(D-2)\Delta_{p+1}}}\right] \\ &+ \epsilon_{p+1}\mathcal{C}^{-1}H_{p+1}^{-\frac{4(p+2)}{(D-2)\Delta_{p+1}}}f=0,\end{aligned} \tag{27}$$

where $\mathcal{C}=\left(\frac{\Delta+2}{\Delta}Qz\right)^{\frac{4}{(D-2)(\Delta+2)}}$ is the conformal factor in the metric (23), $z=z(\lambda)$ for the null geodesic motion is constant or is given by Eq. (25), and $\epsilon_{p+1}=0,1$ respectively for the massless and the massive test particle. On the other hand, for the test particle in the source background (6) of the $(D-1)$-dimensional p-brane, the geodesic motion is described by

$$\left(\frac{dx}{d\lambda}\right)^2+\left[\epsilon_p H_p^{-\frac{4(p+1)}{(D-3)\Delta_p}}+\frac{\mathcal{J}^2}{H_p^{\frac{8(p+1)}{(D-3)\Delta_p}}x^2}\right]f-E^2H_p^{\frac{4(D-2p-5)}{(D-3)\Delta_p}}=0, \tag{28}$$

where $\epsilon_p=1,0$ respectively for the timelike and the null geodesic. Here, $\mathcal{J}$ in the above is defined in terms of the conserved angular momenta J^m of the test particle as

$$\mathcal{J}^2\equiv\sum_{m=1}^{[\frac{D-p-2}{2}]}\frac{(J^m)^2}{\mu_m^2}, \tag{29}$$

where the direction cosines μ_m specifying the direction of x are constant due to conservation of the direction of angular momentum (therefore $\mathcal{J}$ is also constant).

First, when the $(p+1)$-brane is uncharged (i.e., $H_{p+1}=1$), one can bring the equation (27) for the null geodesic motion ($\epsilon_{p+1}=0$) to the form of the equation (28) for the time-like geodesic motion ($\epsilon_p=1$) in the $(D-1)$-dimensional uncharged p-brane background. To see this, we consider the following equation obtained from Eq. (27) by setting $\epsilon_{p+1}=0$ and $H_{p+1}=1$, and substituting (25):

$$\begin{aligned}\left(\frac{dx}{d\lambda}\right)^2&+\left(\frac{\Delta+2}{\Delta}Qz_0\right)^{-\frac{8}{(D-2)(\Delta+2)}}\lambda^{-\frac{8}{(D-2)(\Delta+2)+4}}\left[\left\{\left(\frac{\Delta+2}{\Delta}Qz_0\right)^{\frac{8}{(D-2)(\Delta+2)}}\right.\right.\\&\left.\left.\times\left(\frac{(D-2)(\Delta+2)z_0}{(D-2)(\Delta+2)+4}\right)^2+\frac{\mathcal{J}^2}{x^2}\right\}\left(1-\frac{\mu}{x^{D-p-4}}\right)-E^2\right]=0.\end{aligned} \tag{30}$$

By redefining the radial coordinate, constants of the motion and parameters in the following way:

$$\begin{aligned}\tilde{x}&\equiv\mathcal{A}x,\qquad \tilde{E}\equiv\mathcal{A}E,\qquad \tilde{J}\equiv\mathcal{A}^2J,\qquad \tilde{\mu}\equiv\mathcal{A}^{D-p-4}\mu,\\ \tilde{\lambda}&\equiv\frac{(D-2)(\Delta+2)+4}{(D-2)(\Delta+2)}\left(\frac{\Delta+2}{\Delta}Qz_0\right)^{-\frac{2}{(D-2)(\Delta+2)}}\lambda^{\frac{(D-2)(\Delta+2)}{(D-2)(\Delta+2)+4}},\end{aligned} \tag{31}$$

where

$$\mathcal{A}\equiv\left(\frac{(D-2)(\Delta+2)z_0}{(D-2)(\Delta+2)+4}\right)^{-1}\left(\frac{\Delta+2}{\Delta}Qz_0\right)^{-\frac{4}{(D-2)(\Delta+2)}}, \tag{32}$$

one can bring Eq. (30) into the following form:

$$\left(\frac{d\tilde{x}}{d\tilde{\lambda}}\right)^2+\left(1+\frac{\tilde{\mathcal{J}}^2}{\tilde{x}^2}\right)\left(1-\frac{\tilde{\mu}}{\tilde{x}^{D-p-4}}\right)=\tilde{E}^2. \tag{33}$$

This reproduces the equation for the timelike geodesic motion in the background of uncharged $(D-1)$-dimensional p-brane, i.e., Eq. (28) with $\epsilon_p = 1$ and $H_p = 1$. This result generalizes the result of Ref. [3] to the case of an uncharged black brane in a dilatonic domain wall in arbitrary spacetime dimensions.

Next, we consider the charged branes. The equation for the null geodesic motion (with nontrivial lightlike motion along the z-direction) in the background of the brane-world charged $(p+1)$-brane, i.e., Eq. (27) with $\epsilon_{p+1} = 0$ and Eq. (25) substituted, reduces to the following form after the quantities are redefined as in Eq. (31):

$$\left(\frac{d\tilde{x}}{d\tilde{\lambda}}\right)^2 + \left[\tilde{H}_{p+1}^{-\frac{4}{\Delta_{p+1}}} - \frac{\tilde{\mathcal{J}}^2}{\tilde{H}_{p+1}^{\frac{8(p+2)}{(D-2)\Delta_{p+1}}}\tilde{x}^2}\right]\tilde{f} - \tilde{E}^2\tilde{H}_{p+1}^{\frac{4(D-2p-6)}{(D-2)\Delta_{p+1}}} = 0, \tag{34}$$

where

$$\tilde{H}_{p+1} = 1 + \frac{\mu \sinh^2\delta_{p+1}}{(\mathcal{A}\tilde{x})^{D-p-4}}, \qquad \tilde{f} = 1 - \frac{\tilde{\mu}}{\tilde{x}^{D-p-4}}. \tag{35}$$

This equation is different from the equation for the timelike geodesic in the background of the $(D-1)$-dimensional p-brane, i.e., Eq. (28) with $\epsilon_p = 1$, since the powers of the harmonic functions are different in the two equations. This difference might be attributed to the fact that when one compactifies the Einstein-frame metric for the D-dimensional $(p+1)$-brane (i.e., Eq. (23) without the z-dependent conformal factor) along one of its longitudinal directions (i.e., the z-direction) by using the KK metric Ansatz without the Weyl-scaling factor in the $(D-1)$-dimensional part of the metric (i.e., $g_{\mu\nu} = \mathrm{diag}(\bar{g}_{\bar{\mu}\bar{\nu}}, \bar{\varphi})$ with $\mu, \nu = 0, 1, ..., D-1$ and $\bar{\mu}, \bar{\nu} = 0, 1, ..., D-2$), one gets non-Einstein-frame metric for the $(D-1)$-dimensional p-brane. However, as can be seen from Eq. (26), even in such non-Einstein-frame spacetime in $D-1$ dimensions, the equation for the timelike geodesic will look different because of the dependence of the (z, z)-component of the D-dimensional metric on x. (Cf. The second term on the LHS of Eq. (26) for the null geodesic motion in D dimensions is identified with the last term on the LHS of Eq. (26) for the timelike geodesic motion in $D-1$ dimensions.) We just write down the equation for the timelike geodesic motion in such $(D-1)$-dimensional background for comparison:

$$\left(\frac{dx}{d\lambda}\right)^2 + \left[H_{p+1}^{-\frac{4(p+2)}{(D-2)\Delta_{p+1}}} + \frac{\mathcal{J}^2}{H_{p+1}^{\frac{8(p+2)}{(D-2)\Delta_{p+1}}}x^2}\right]f - E^2 H_{p+1}^{\frac{4(D-2p-6)}{(D-2)\Delta_{p+1}}} = 0. \tag{36}$$

As mentioned, the first terms in the square brackets of Eqs. (34) and (36) are different. So, only when the probe motion along the z-direction is trivial, i.e., z = constant, the null geodesic motion in the background of the D-dimensional $(p+1)$-brane (described by Eq. (27) with $\epsilon_{p+1} = 0$ and z = constant) reproduces the null geodesic motion in the background of the p-brane in one lower dimensions (described by Eq. (28) with $\epsilon_p = 0$). In the case of the uncharged branes, the above problems did not arise because the (z, z)-component of the D-dimensional metric is independent of x and the dimensional reduction of the Einstein-frame metric for the D-dimensional uncharged $(p + 1)$-brane along a longitudinal direction (without the Weyl-scaling term in the metric) leads to the Einstein-frame metric for the $(D - 1)$-dimensional uncharged p-brane. It is interesting to note that when there is no constraint on the dilaton coupling parameters, i.e. the $p = D - 4$ case (Cf. see Eq. (5)), the equation (34) for the null geodesic motion (with nontrivial motion along the z-direction) in the background of the brane-world $(p+1)$-brane reproduces the equation for the timelike geodesic motion in the background of the p-brane in one lower dimensions, i.e., Eq. (28) with $\epsilon_p = 1$ and Eq. (36). In such case, the transverse (to the domain wall) component of the metric (23), i.e., the (z, z)-component, is independent of the longitudinal coordinates of the domain wall. This is in accordance with our speculation on the source of disparity in the probe particle dynamics in the backgrounds of *charged* branes.

In fact, one of the assumptions of the RS model is that the D-dimensional conformally flat metric (of the domain wall) should have the perturbation around the flat metric along the longitudinal directions of the domain wall, only. Indeed, as mentioned in the above, one can see from Eq. (26) that the extra space component of the metric, i.e., g_{zz}, should be independent of the longitudinal coordinates of the domain wall, in order for the null geodesic motion (with non-trivial lightlike motion along the z-direction) in D dimensions to reproduce the timelike geodesic motion in the background of the source p-brane in $D - 1$ dimensions. So, for the $p = D - 4$ case, even if the branes are charged, the null geodesic motion in the source background of the brane-world $(p+1)$-brane reproduces the timelike geodesic motion in the source background of the p-brane in one lower dimensions, because the extra space component g_{zz} of the brane-world $(p+1)$-brane metric is independent of the longitudinal coordinates of the domain wall. Also, recently, it is shown [22] that the metric perturbation in the direction transverse to the domain wall is not localized on the brane. On the other hand, in general, the extra space component g_{zz} of the spacetime metrics for charged brane solutions in the domain wall depends on the longitudinal coordinates $\mathbf{x}$ of the domain wall. So, it seems inevitable that all the charged branes in brane worlds do not reproduce physics in one lower dimensions. This might be the indication of either the need to modify physics in one lower dimensions (which seems unlikely) or the limitation of the current RS model that needs to be modified so that it can accommodate, for example, charged black holes or branes that will reproduce lower-dimensional physics. (Also, the non-dilatonic domain wall of the RS model in Refs. [11, 1, 12] does not admit even charged black string solutions, which are supposed to be identified as charged black holes in four dimensions, because of the constraint on the dilaton coupling parameters.)

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